Improving Reachability in Vector Addition Systems Through Pumpability

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Vector Addition Systems

d-dimensional vector addition system (d-VAS)

=

d counters + finitely many updates

Formal Definition.

- Transition: $t \in \mathbb{Z}^d$
- Configuration: $\boldsymbol{x} \in \mathbb{N}^d$
- Step: $\boldsymbol{x} \xrightarrow{\boldsymbol{u}} \boldsymbol{x} + \boldsymbol{u}$ if $\boldsymbol{u} \in T$ and $\boldsymbol{x} + \boldsymbol{u} \in \mathbb{N}^d$

A run from $\boldsymbol{s}$ to $\boldsymbol{t}$ is a sequence of transitions $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ such that

$$\boldsymbol{s} \xrightarrow{\boldsymbol{u}_1} \boldsymbol{x}_1 \xrightarrow{\boldsymbol{u}_2} \boldsymbol{x}_2 \xrightarrow{\boldsymbol{u}_3} \ldots \xrightarrow{\boldsymbol{u}_k} \boldsymbol{t}$$

We write $\boldsymbol{s} \xrightarrow{*} \boldsymbol{t}$ if there is a run from $\boldsymbol{s}$ to $\boldsymbol{t}$.

Reachability Problem in d-VAS.

Input: A d-VAS V and two configurations $\boldsymbol{s}, \boldsymbol{t}$.

Question: Is $\boldsymbol{s} \xrightarrow{*} \boldsymbol{t}$ holds in V?

Vector Addition Systems with States

Add State Control.

$$\boldsymbol{x} \in \mathbb{N}^d \implies q(\boldsymbol{x}) := (q, \boldsymbol{x}) \in Q \times \mathbb{N}^d,$$

$$T \subseteq \mathbb{Z}^d \implies T \subseteq Q \times \mathbb{Z}^d \times Q.$$

d-dimensional vector addition system with states (d-VASS)

=

d counters + finite state automata

All the counters should be non-negative during the run.

$$p(\boldsymbol{x}) \xrightarrow{t} q(\boldsymbol{y}) \text{ if } t = (p, \boldsymbol{y} - \boldsymbol{x}, q) \in T \text{ and } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^d.$$

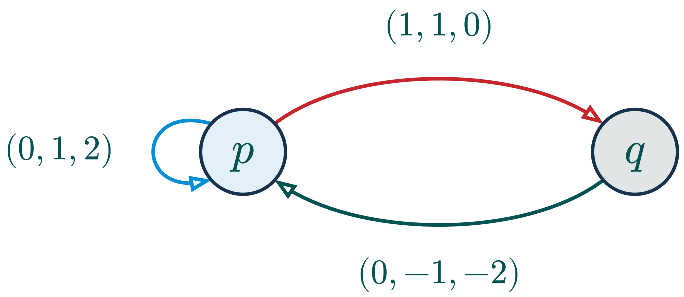
Fu, Yang and Zheng, 2024; Czerwinski, Jecker, Lasota, Leroux and Orlikowski, 2023.

Reachability in d-VASS in $\mathsf{F}_{\frac{d}{2}+\mathcal{O}(1)}$-hard and in F_d.

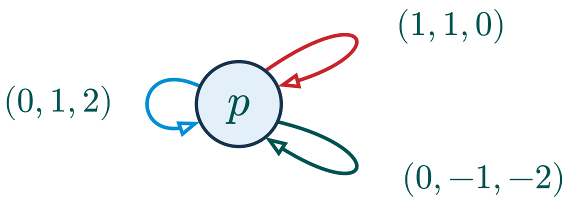
The class $(\mathsf{F}_d)_{d \in \mathbb{N}}$ is the fast-growing complexity hierarchy. Please refer to [Schmitz, 2016] for more details.

NOTE: We focus on the reachability problem in d-VAS in this work.

Motivation: Why VAS Might Be Easier



multiple states v.s. single state



d-VAS = d-VASS with only one state

> Fu, Yang and Zheng, 2024.
> Reachability in d-VASS is in F_d for $d \geq 3$.

$\Longrightarrow$ Reachability in d-VAS is in F_d for $d \geq 3$.

> Czerwinski, Jecker, Lasota, and Orlikowski, 2025.
> Reachability in strongly connected 3-VASS is in **PSPACE**.

$\Longrightarrow$ Reachability in 3-VAS is in **PSPACE**.

A VAS is always strongly connected.

A VASS is strongly connected if its underlying directed graph is strongly connected.

NEXT: Can the one-state structure buy us a better fixed-dimensional bound?

Two Dimensions Better

Hopcroft and Pansiot, 1979.

- A d-VASS can be simulated by a $(d+3)$-VAS with polynomial overhead.
- 5-VAS have effectively semilinear reachability sets, whereas 3-VASS do not.

5-VAS may genuinely be simpler than 3-VASS.

This leaves a natural 3-dimensional gap between the two models in fixed dimension.

This Work.

- Reachability in $(d+2)$-VAS is in F_d for every $d \geq 3$.
- Reachability in 5-VAS is in **ELEMENTARY**.
- Reachability in 4-VAS is in **PSPACE** under binary encoding and is **NL**-complete under unary encoding.

NEXT: Where do the two dimensions go?

Geometric Dimension

KEY POINT: We lower the component dimension, not necessarily the number of counters d.

Following [Leroux and Schmitz, 2019], geometric dimension is defined through cycle spaces.

Cycle Space.

$$\textsc{cyc}(V) := \mathsf{span}(\{\Delta(\theta) \mid \theta \text{ is a simple cycle in } V\}).$$

Two Geometric Dimensions.

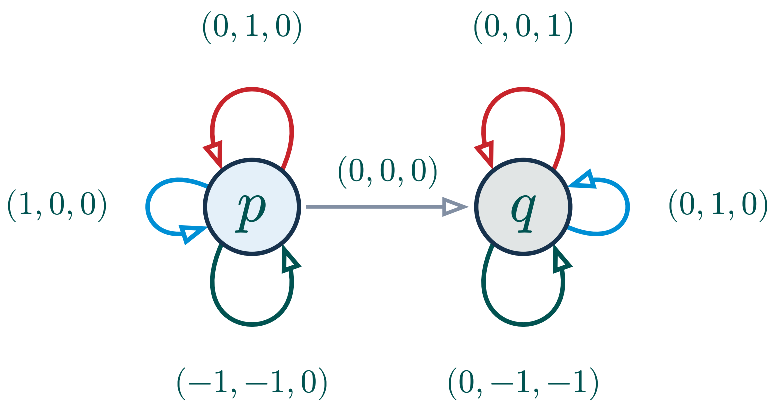
- Cycle-Dimension:

$$\dim_{\mathsf{cyc}}(V) := \dim(\textsc{cyc}(V)).$$

- Component-Dimension:

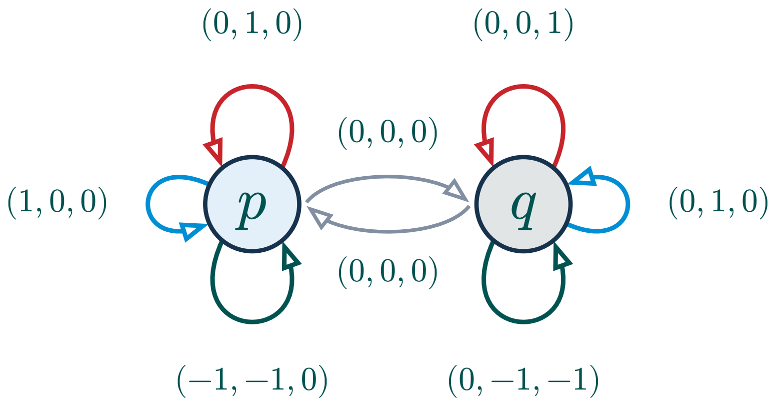
$$\dim_{\mathsf{com}}(V) := \max\big\{\dim_{\mathsf{cyc}}(C) \mid C \text{ is a SCC of } V\big\}.$$

Geometric Dimension



$$\dim_{\mathsf{cyc}}(V_{p\to q}) = \dim(\mathbb{Q}^3) = 3$$

$$\dim_{\mathsf{com}}(V_{p\to q}) = \max(\dim_{\mathsf{cyc}}(V_p)), \dim_{\mathsf{cyc}}(V_q))) = 2$$



$$\dim_{\mathsf{cyc}}(V_{p\leftrightarrow q}) = \dim(\mathbb{Q}^3) = 3$$

$$\dim_{\mathsf{com}}(V_{p\leftrightarrow q}) = \dim_{\mathsf{cyc}}(V_{p\leftrightarrow q}) = 3$$

Length-Boundedness

We write $\textsc{len}(V, s, t)$ for the set of lengths of runs from s to t.

Fu, Yang and Zheng, 2024. Let (V, s, t) be a VASS of component-dimension $m \geq 3$. If $\textsc{len}(V, s, t) \neq \varnothing$, then

$$\min \textsc{len}(V, s, t) \leq F_m(g(N))$$

where N is the unary size $\textsc{size}(V, s, t)$ and g is some elementary function independent of m, and (V, s, t).

Roadmap.

F_m length bound for VASS with $\dim_{\mathsf{com}}(\cdot) = m$

+

Reduction from $(d+2)$-VAS to VASS with $\dim_{\mathsf{com}}(\cdot) = d$

$\Longrightarrow$ F_d length bound for $(d+2)$-VAS $\overset{*}{\Longrightarrow}$ $(d+2)$-VAS $\in \mathsf{F}_d$

* By a brute-force search, we can enumerate and check all runs of length at most $F_d(g(n))$ in nondeterministic F_d space.

Length-Boundedness

Roadmap.

F_m length bound for VASS with $\dim_{\text{com}}(\cdot) = m$

+

Reduction from $(d+2)$-VAS to VASS with $\dim_{\text{com}}(\cdot) = d$

$\implies$ F_d length bound for $(d+2)$-VAS $\implies$ $(d+2)$-VAS $\in \mathsf{F}_d$

Given a function f, a VASS (V, s, t) is said to be length-bounded by a VASS (V', s', t') with f-amplification, if

$$\min \textsc{len}(V, s, t) \leq \min \textsc{len}(V', s', t'), \qquad \textsc{size}(V', s', t') \leq f(\textsc{size}(V, s, t))$$

and $s \xrightarrow{*} t \text{ in } V \text{ if and only if } s' \xrightarrow{*} t' \in V'$.

Theorem 1. Every $(d+2)$-VAS $(V, \boldsymbol{s}, \boldsymbol{t})$ is length-boounded by a VASS (V', s', t') of component-dimension at most d with polynomial amplification.

NEXT: These bounds may depending on d exponentially, but they are still polynomial in the instance size N.

Wide Sequential VAS

Cone

Recall that $\textsc{cyc}(V) := \mathsf{span}(\{\Delta(\theta) \mid \theta \text{ is a simple cycle in } V\})$.

Cone.

- Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$. Define the cone of X as the set of all non-negative linear combinations of X, i.e.,

$$\textsc{cone}(X) := \left\{\sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \mid \lambda_i \in \mathbb{Q}_{\geq 0}\right\}.$$

- The cone of a VASS V is defined as

$$\textsc{cone}(V) := \textsc{cone}(\{\Delta(\theta) \mid \theta \text{ is a simple cycle in } V\}) \subseteq \textsc{cyc}(V).$$

A cycle can only be repeated for non-negative times.

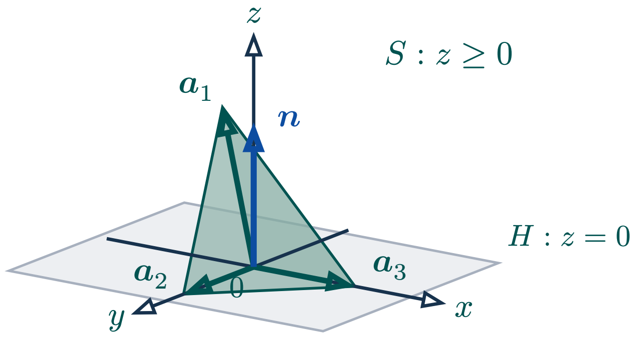
$$\Downarrow$$

$\textsc{cone}(V)$ better describes the behavior of V than $\textsc{cyc}(V)$.

Sequential VAS

A VASS V is said to be wide if $\textsc{cone}(V) = \textsc{cyc}(V)$, and non-wide otherwise.

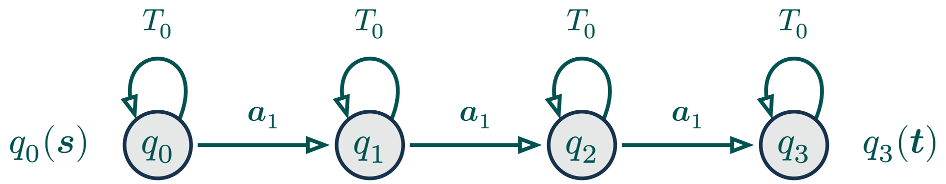
Non-Wide Case.



$\boldsymbol{s} = (0,0,0)$, $\boldsymbol{t} = (0,0,3)$

A run from $\boldsymbol{s}$ to $\boldsymbol{t}$ can use the cycle $\boldsymbol{a}_1$ for exactly 3 times.

$\Downarrow$ Let $T_0 := \{\boldsymbol{a}_2, \boldsymbol{a}_3\}$.



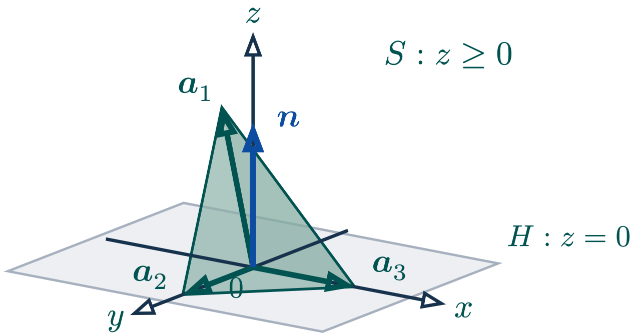
Sequential VAS. Let V be a d-VAS and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k \in \mathbb{Z}^d$ be k vectors. A sequential VAS $S = V[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k]$ is obtained by copying the VAS V for $k+1$ times, and connecting the copies sequentially with transitions $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$.

$$S = V_0[\boldsymbol{a}_1, \boldsymbol{a}_1, \boldsymbol{a}_1] \text{ where } V_0 = T_0 \text{ and } \dim_{\mathsf{cyc}}(V_0) = \dim_{\mathsf{com}}(V_0) = 2$$

Sequential VAS

A VASS V is said to be wide if $\textsc{cone}(V) = \textsc{cyc}(V)$, and non-wide otherwise.

Non-Wide Case.



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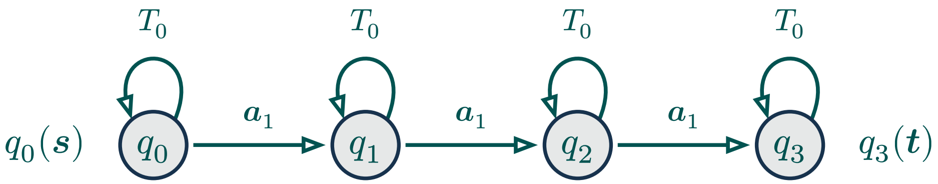


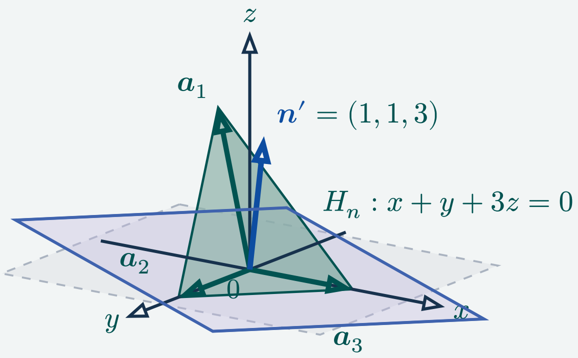
One-state structure $\Longrightarrow$ Repeated one-state structure.

V_0 is still non-wide since $\textsc{cone}(V_0) \subsetneq \textsc{cyc}(V_0)$ in this case.

Can we do better?

Wide Sequential VAS

Choose a Better Separating Hyperplane.



Consider $\boldsymbol{s} = (0,0,0)$, $\boldsymbol{t} = (0,0,3)$ and $\boldsymbol{n}' = (1,1,3)$.

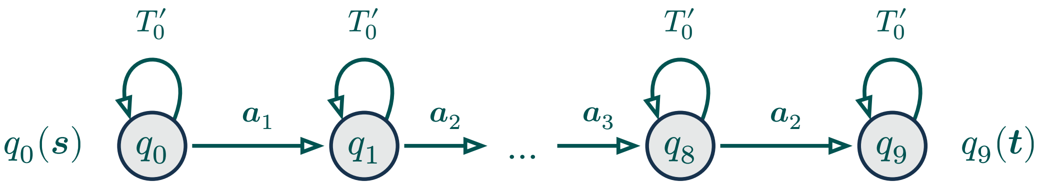
$$\langle \boldsymbol{n}', \boldsymbol{s} \rangle = 0 \le 9 = \langle \boldsymbol{n}', \boldsymbol{t} \rangle,$$

$$\langle \boldsymbol{n}', \boldsymbol{a}_1 \rangle = \langle \boldsymbol{n}', \boldsymbol{a}_2 \rangle = \langle \boldsymbol{n}', \boldsymbol{a}_3 \rangle = 1$$

A run from $\boldsymbol{s}$ to $\boldsymbol{t}$ can use exactly 9 transitions.

Guess a transitions sequence of length 9.

$\xRightarrow{\text{Let } T_0' := \varnothing.}$



$S' = V_0'[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_2]$ where $V_0' = \varnothing$ with $\dim_{\mathsf{cyc}}(V_0) = \dim_{\mathsf{com}}(V_0) = 0$

NEXT: Now S' is a wide sequential VAS. But how to bound the numbers of bridges?

From Arbitrary VAS to Wide Sequential VAS

We can always find a good separating hyperplane, which can be described in polynomial size

Lemma 2. Let $X \subseteq \mathbb{Z}_d$ be a finite set such that $\textsc{cone}(X) \neq \mathsf{span}(X)$. There is a non-zero vector $\boldsymbol{n} \in \mathbb{Z}_d$ such that

1. $\langle \boldsymbol{n}, \boldsymbol{x} \rangle \geq 0$ for all $\boldsymbol{x} \in X$;
2. let $X_0 := \{\boldsymbol{x} \in X \mid \langle \boldsymbol{n}, \boldsymbol{x} \rangle = 0\}$; then $\textsc{cone}(X_0) = \mathsf{span}(X_0)$;
3. $\|\boldsymbol{n}\| \leq (d+1) \cdot (r\|X\|)^r$, where $r := \dim(\mathsf{span}(X))$.

Proof sketch.

- Existence: By the Farkas-Minkowski-Weyl Theorem, $\textsc{cone}(X)$ can be described by $A\boldsymbol{x} \geq \boldsymbol{0}$ for some integer matrix A. Take $\boldsymbol{n}$ be the sum of all the rows of A.
- Size: By [Gathen and Sieveking, 1978], we can find a non-zero integer solution $\boldsymbol{n}$ subject to the constraints with polynomially bounded norm.

□

From Arbitrary VAS to Wide Sequential VAS

We can always find a good separating hyperplane, which can be described in polynomial size

Lemma 4. Let $X \subseteq \mathbb{Z}_d$ be a finite set such that $\textsc{cone}(X) \neq \mathsf{span}(X)$. There is a non-zero vector $\boldsymbol{n} \in \mathbb{Z}_d$ such that

1. $\langle \boldsymbol{n}, \boldsymbol{x} \rangle \geq 0$ for all $\boldsymbol{x} \in X$;
2. let $X_0 := \{\boldsymbol{x} \in X \mid \langle \boldsymbol{n}, \boldsymbol{x} \rangle = 0\}$; then $\textsc{cone}(X_0) = \mathsf{span}(X_0)$;
3. $\|\boldsymbol{n}\| \leq (d+1) \cdot (r\|X\|)^r$, where $r := \dim(\mathsf{span}(X))$.

Lemma 5.

- Every d-VAS $(V, \boldsymbol{s}, \boldsymbol{t})$ is length-bounded by a wide sequential d-VAS (S, s', t') with polynomial amplification.
- Moreover, if V is non-wide, then $\dim_{\mathsf{cyc}}(S) \leq \dim_{\mathsf{cyc}}(V) - 1$.

If V is itself wide, then $S = V$ and $\dim_{\mathsf{cyc}}(S) = \dim_{\mathsf{cyc}}(V)$.

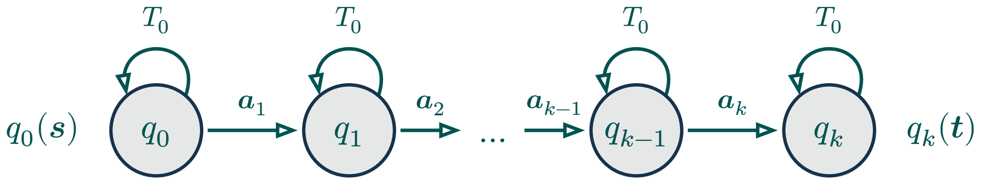
NEXT: It suffices to prove Theorem 1 for wide sequential VAS.

Pumping Technique

Fixed Coordinates

For a sequential d-VAS $S = V_0[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k]$, we say that a coordinate $i \in [d]$ is fixed in S if $\boldsymbol{u}(i) = 0$ holds for every transition $\boldsymbol{u}$ in V.

Assume that the i-th coordinate is fixed in S.



$$q_0 \mapsto \boldsymbol{s}(i) \quad \mapsto \boldsymbol{s}(i) + \boldsymbol{a}_1(i) \quad \cdots \quad q_{k-1} \mapsto \boldsymbol{s}(i) + \sum_{j=1}^{k-1} \boldsymbol{a}_j(i) \quad q_k \mapsto \boldsymbol{t}(i).$$

The i-th coordinate can be derived from the state. $\Longrightarrow$ Remove the i-th coordinate from S.

NEXT: We only consider wide sequential VAS with no fixed coordinates.

Pumpability

Pumpability is another important notion in the literature.

Pumpability.

- A configuration $q(\boldsymbol{x})$ is forward pumpable if $q(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{x}')$ in V with $\boldsymbol{x}' \geq \boldsymbol{x} + \mathbf{1}$.
- A configuration $q(\boldsymbol{x})$ is backward pumpable if $q(\boldsymbol{x}') \xrightarrow{*} q(\boldsymbol{x})$ in V with $\boldsymbol{x}' \geq \boldsymbol{x} + \mathbf{1}$.

A $\mathbb{Z}$-run $p(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Z}} q(\boldsymbol{y})$ is defined similar as a run but without the non-negativity requirement.

Lemma 6. Let $S = V[\boldsymbol{a}_1, ..., \boldsymbol{a}_k]$ be a wide sequential d-VAS with source state p and target state q. Let $p(\boldsymbol{x}) \xrightarrow{\pi}_{\mathbb{Z}} q(\boldsymbol{y})$ be a $\mathbb{Z}$-run in S, where $p(\boldsymbol{x})$ is forward pumpable and $q(\boldsymbol{y})$ is backward pumpable. Then $p(\boldsymbol{x}) \xrightarrow{\rho} q(\boldsymbol{y})$ for some ρ in S with $|\rho| \leq \text{poly}(N)$.

Pumpability $+$ Wideness + $\mathbb{Z}$-Reachability $\Longrightarrow$ Polynomially Short Legal Run.

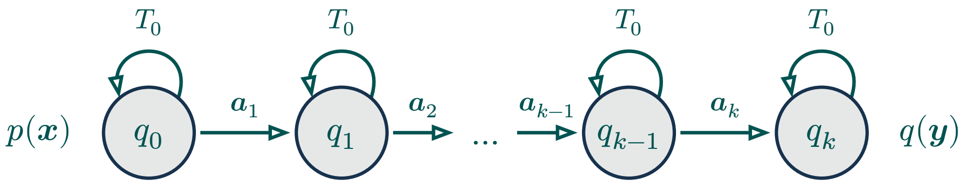
Shortening a $\mathbb{Z}$-Run

We establish a slightly stronger version of the result for $\mathbb{Z}$-run wide sequential VAS.

A $\mathbb{Z}$-run $p(\boldsymbol{x}) \xrightarrow{*}_{\mathbb{Z}} q(\boldsymbol{y})$ is defined similar as a run but without the non-negativity requirement.

Lemma 7. Let $S = V[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k]$ be a wide sequential d-VAS with source state p and target state q. Let $p(\boldsymbol{x}) \xrightarrow{\pi}_{\mathbb{Z}} q(\boldsymbol{y})$ be a $\mathbb{Z}$-run in S, where $p(\boldsymbol{x})$ is forward pumpable and $q(\boldsymbol{y})$ is backward pumpable. Then $p(\boldsymbol{x}) \xrightarrow{\rho} q(\boldsymbol{y})$ for some ρ in S with $|\rho| \leq \text{poly}(N)$.

The $\mathbb{Z}$-run π can be shortened to a $\mathbb{Z}$-run ς.



- Let $\boldsymbol{\lambda}_i = \#\boldsymbol{u}_i$ in π for each transition $\boldsymbol{u}_i$ in T_0.

$$\boldsymbol{x} + T_0 \cdot \boldsymbol{\lambda} + \sum_{i=1}^{k} \boldsymbol{a}_i = \boldsymbol{y}$$

- By [Gathen and Sieveking, 1978], a small non-negative integer solution $\boldsymbol{\lambda}'$ exists with $|\boldsymbol{\lambda}'| \leq \text{poly}(N)$.
- $\boldsymbol{\lambda}'$ induces a $\mathbb{Z}$-run ς with $|\varsigma| \leq \text{poly}(N)$.

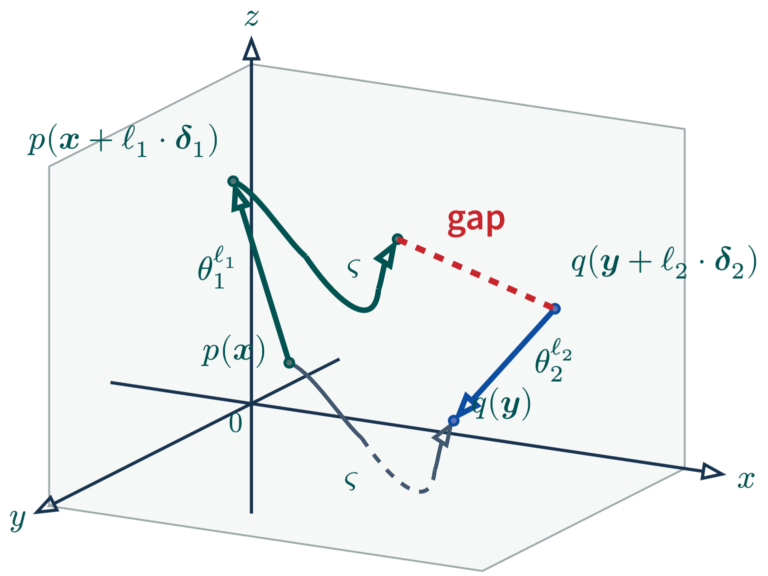
Lifting a $\mathbb{Z}$-Run

There exist short pumping cycles.

By [Rackoff, 1978], the shortest covering runs are bounded by poly(N) for fixed d.

- $\boldsymbol{x} + \mathbf{1}$ is coverable from $p(\boldsymbol{x}) \implies$ There exists a cycle $p(\boldsymbol{x}) \xrightarrow{\theta_1} p(\boldsymbol{x} + \boldsymbol{\delta}_1)$ with $|\theta_1| \leq \text{poly}(N)$ and $\boldsymbol{\delta}_1 \geq \mathbf{1}$.
- $\boldsymbol{y} + \mathbf{1}$ is backward coverable from $q(\boldsymbol{y}) \implies$ There exists a cycle $q(\boldsymbol{y} + \boldsymbol{\delta}_2) \xrightarrow{\theta_2} q(\boldsymbol{y})$ with $|\theta_2| \leq \text{poly}(N)$ and $\boldsymbol{\delta}_2 \geq \mathbf{1}$.

Leverage pumpability to lift the $\mathbb{Z}$-run ς to a legal run ρ.



- Forward pumping: Choose $\ell_1 \in \mathbb{N}$ such that $\boldsymbol{x} + \ell_1 \cdot \boldsymbol{\delta}_1 \geq |\varsigma| \cdot N$.

$$p(\boldsymbol{x}) \xrightarrow{\theta_1^{\ell_1}} p(\boldsymbol{x} + \ell_1 \cdot \boldsymbol{\delta}_1) \xrightarrow{\varsigma} q(\boldsymbol{y} + \ell_1 \cdot \boldsymbol{\delta}_1).$$

- Backward pumping: Choose $\ell_2 \in \mathbb{N}$ such that $\ell_1 \cdot \boldsymbol{\delta}_1 = \ell_2 \cdot \boldsymbol{\delta}_2$.

$$q(\boldsymbol{y} + \ell_2 \cdot \boldsymbol{\delta}_2) \xrightarrow{\theta_2^{\ell_2}} q(\boldsymbol{y}).$$

- Note: If $\ell_1, \ell_2 > 0$, then $\ell_1 \cdot \boldsymbol{\delta}_1 \neq \ell_2 \cdot \boldsymbol{\delta}_2$ in general!

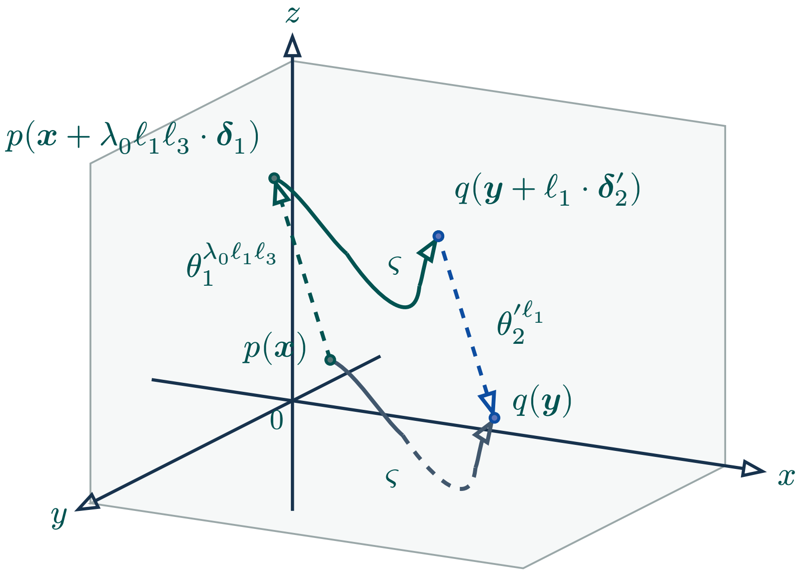
Aligning the Pumping Directions

Compensate the gap between by wideness.

- Recall that S is wide, which means that

$$\delta_2 - \delta_1 \in \textsc{cyc}(V) = \textsc{cone}(V) \xRightarrow{\text{Carathéodory's Theorem}} \lambda_0 \cdot (\delta_2 - \delta_1) = \sum_{i=1}^{d} \lambda_i \cdot \boldsymbol{u}_i,$$

where $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d \in T_0$, $\lambda_0, \ldots, \lambda_d \geq 0$ and $\lambda_0 \neq 0$. This equation system has a non-zero minimal solution λ^*.



$$\text{Let } o := \boldsymbol{u}_1^{\lambda_1^*} \cdots \boldsymbol{u}_d^{\lambda_d^*}$$

$$\Downarrow$$

$$\text{Let } \theta_2' := o^{\ell_3}\theta_2^{\lambda_0\ell_3} \text{ where } \boldsymbol{y} + \ell_3 \cdot \boldsymbol{\delta}_2 \geq |o| \cdot N$$

$$\Downarrow$$

$$q(\boldsymbol{y} + \lambda_0\ell_3 \cdot \boldsymbol{\delta}_1) \xrightarrow{\theta_2'} q(\boldsymbol{y}) \text{ where } \theta_2' := o^{\ell_3}\theta_2^{\lambda_0\ell_3}$$

$$\Downarrow$$

$$\text{Let } \boldsymbol{\delta}_2' := \lambda_0\ell_3 \cdot \boldsymbol{\delta}_1, \text{and } |\boldsymbol{\delta}_2'| \leq \mathsf{poly}(N).$$

Rackoff's Theorem

KEY POINT: What about the non-pumpable case?

Pumpablity can be extracted in a strongly connected VASS by using Rackoff's Lemma.

Lemma 8. For each $U \in \mathbb{N}$ and $p(\boldsymbol{x}) \xrightarrow{\pi} q(\boldsymbol{y})$ in a d-VASS V, there exists $B = \text{poly}(U, N)$ such that

$$\Delta\Big(p(\boldsymbol{x}) \xrightarrow{\pi} q(\boldsymbol{y})\Big) \quad = \quad \Delta\Big(p(\boldsymbol{x}) \xrightarrow{\rho} q(\boldsymbol{y}')\Big) + \Delta(\text{several simple cycles}),$$

where $|\rho| \leq B$ such that its target $q(\boldsymbol{y}')$ satisfies $\boldsymbol{y}'(i) \geq U$ for all $i \in \textsc{ub}\ (\pi, B)$.

Let $\textsc{ub}\ (\pi, B) := \{i \in [d] : \boldsymbol{z}(i) \geq B \ \text{ for some } \boldsymbol{z} \text{ on } \ \pi\}$, i.e., the coordinates that ever reached a value above B along π.

Non-pumpable $\Longrightarrow$ A bounded coordinate $i \in [d]$ exists with one SCC

$\Longrightarrow$ Encode the bounded coordinate into the state space $(q, \boldsymbol{x}(i))$

$\Longrightarrow$ **Dimension Reduction by One**

New Pumping Technique

We show that pumpability is guaranteed if all but one coordinates can be raised above some large value in a run.

Theorem 9. Let S be a wide sequential d-VAS and $s \xrightarrow{\pi} t$ be a run in S. There exists a polynomial $B = \text{poly}(N)$ such that if $|\textsc{ub}(\pi, B)| \geq d - 1$, then there is a run ρ such that $|\rho| \leq B$ and

$$s \xrightarrow{\rho} c \xrightarrow{*}_{\mathbb{Z}} t$$

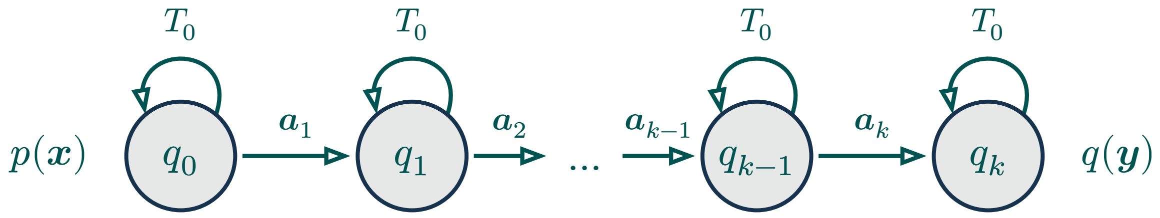
for some c forward pumpable in S.

Non-pumpable $\Longrightarrow$ Two bounded coordinate

$\Longrightarrow$ **Dimension Reduction by Two**

New Pumping Technique

Proof sketch. Let $U := (1+N)^2$ and B be the bound given by Lemma 7. Assume that $\textsc{ub}(\pi, B) = [d-1]$.



I. Obtain a configuration with $d-1$ large values.

By Lemma 7, we can construct a run ρ with target configuration $p(\boldsymbol{z})$ such that $\boldsymbol{z}(i) \geq U$ for all $i \in [d-1]$, $|\rho| \leq B$, and

$$\Delta\left(s \xrightarrow{\pi} t\right) \quad = \quad \Delta\left(s \xrightarrow{\rho} c := p(\boldsymbol{z})\right) + \Delta(\text{several simple cycles}).$$

- Repeated one state structure: Simple cycles = Transitions in T_0, which can be moved to anywhere in a $\mathbb{Z}$-run.

$$s \xrightarrow{\rho} c \xrightarrow{\text{several simple cycles}}_{\mathbb{Z}} t.$$

New Pumping Technique

II. Pump the last coordinate.

- No fixed coordinates in $S \implies$ There is a self-loop $\boldsymbol{u}$ in T_0 with $\boldsymbol{u}(d) \neq 0$.
- Wideness $\implies$ There is a self-loop $\boldsymbol{u}$ in T_0 with $\boldsymbol{u}(d) > 0$.

Choose $H := N + 1$. Then $\boldsymbol{z}' := \boldsymbol{z} + H \cdot \boldsymbol{u} \geq (U - H \cdot N) \cdot \mathbf{1} \geq (N+1) \cdot \mathbf{1}$.

$$s \xrightarrow{\rho} c = p(\boldsymbol{z}) \xrightarrow{\boldsymbol{u}^H} p(\boldsymbol{z}')$$

III. Show that c is forward pumpable.

Let $o :=$ self-loops in $\boldsymbol{u}^H \rho$ be a new cycle in state p.

$$\Delta(o) + \Delta(\text{bridges in } \rho) = \Delta(\rho \boldsymbol{u}^H) = \boldsymbol{z}' - \boldsymbol{x},$$

$$\implies \Delta(o) \geq \boldsymbol{z}' - \boldsymbol{x} - \Delta(\text{bridges in } \rho) \geq (N+1) \cdot \mathbf{1} - N \cdot \mathbf{1} = \mathbf{1}.$$

$\square$

Dichotomy of Boundedness

Definition 10. We say that π is almost bounded by B if there exists a partition $\pi = \pi_1\pi_2$ such that both

$$|\textsc{ub}(\pi_1, B)|, |\textsc{ub}(\pi_2, B)| \le d - 2$$

Definition 11. We say that π is almost unbounded by B if there exists a partition $\pi = \pi_1\pi_2$ such that

$$|\textsc{ub}(\pi_1, B)|, |\textsc{ub}(\pi_2, B)| \ge d - 1.$$

Note that the partitions in both notions are existentially quantified.

Lemma 12. Suppose that (V, s, t) is a d-VASS with $s \xrightarrow{\pi} t$ in V and $B \in \mathbb{N}$. Then either π is almost unbounded by B, or π is almost bounded by $B + N$.

Almost Unbounded Case

Lemma 13. For any $d \in \mathbb{N}$, there exists a polynomial U_d such that given a wide sequential d-VAS (S, s, t) with $s \xrightarrow{\pi} t$ in S, if π is almost unbounded by $U_d(N)$, then there exists a run $s \xrightarrow{\rho} t$ in S of length $|\rho| \leq U_d(N)$.

Proof Sketch. By definition, assume that $s \xrightarrow{\pi_1} c \xrightarrow{\pi_2} t$ with $|\textsc{ub}(\pi_1, U_d(N))|, |\textsc{ub}(\pi_2, U_d(N))| \geq d-1$ (U_d: TBA).

- By applying Theorem 8 to π_1 and symmetrically to the reverse of π_2,

$$s \xrightarrow{\rho_1} \underbrace{c_1 \xrightarrow{*}_{\mathbb{Z}} c \xrightarrow{*}_{\mathbb{Z}} c_2}_{\textit{merge as a } \mathbb{Z}\textit{-run}} \xrightarrow{\rho_2} t,$$

- By Lemma 6, we can pump c_1 and c_2 to obtain a run ς with

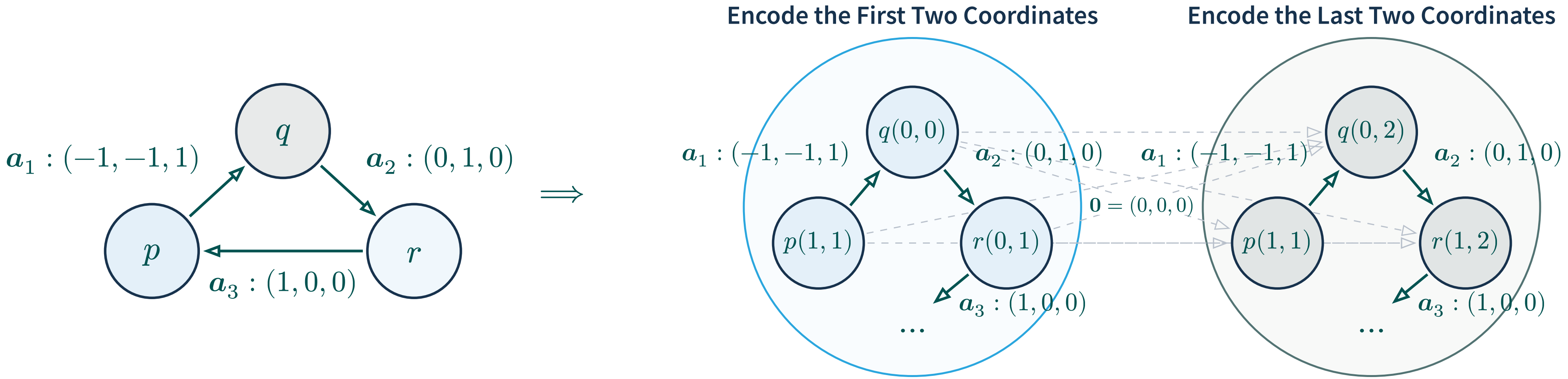
$$s \xrightarrow{\rho_1} c_1 \xrightarrow{\varsigma} c_2 \xrightarrow{\rho_2} t.$$

- Compute the polynomial bound.

□

Almost Bounded Case

Lemma 14. For every $B \in \mathbb{N}$ and d-VASS (V, s, t) with $s \xrightarrow{\pi} t$ in V, if π is almost bounded by B, then (V, s, t) is length-bounded by a d-VASS (V', s', t') with poly(B, N)-amplification, where $dim_{\text{com}}(V') \leq d - 2$.



Component-Dimension: Each SCC has its own bounded coordinates.

$(d+2)$-VAS

$\Downarrow$

wide sequential $(d+2)$-VAS

$\Downarrow$

$$\textbf{Dichotomy} \Longrightarrow \begin{cases} \text{Almost unbounded case} \Longrightarrow \text{Pumpability} \Longrightarrow \text{Polynomially short witness} \\ \text{Almost bounded case} \Longrightarrow \text{VASS with component-dimension at most } d \end{cases}$$

$\Downarrow$

$F_d(N)$ length bound

$\Downarrow$

A F_d-space nondeterministic brute-force algorithm for $(d+2)$-VAS reachability.

Low Dimensional VAS

Finer Characterization

By Theorem 1, we only know that

5-VAS Reachability $\in \mathsf{F}_3$ and 4-VAS Reachability $\in$ **EXPSPACE**.

To estabilish better upper bounds, we need a finer characterization.

Fine-Grained Characterization. For every fixed $d \geq 2$, each d-VAS reachability instance is length-bounded, with polynomial amplification, by a VASS V' in one of the following two forms.

I. Reduce Actual Dimension.

V' is a $(d-2)$-VASS with

$$\dim_{\text{cyc}}(V') \leq \dim_{\text{cyc}}(V) - 1.$$

II. Reduce Compopnent-Dimension.

$V' = (V_1)\mathbf{0}(V_2)$ with

$$\dim_{\text{cyc}}(V_i) \leq \max\big(0, \dim_{\text{cyc}}(V) - 3\big)$$

4-VAS

I. Reduce Actual Dimension.

V' is a $(d-2)$-VASS with

$$\dim_{\text{cyc}}(V') \leq \dim_{\text{cyc}}(V) - 1.$$

II. Reduce Compopnent-Dimension.

$V' = (V_1)\mathbf{0}(V_2)$ with

$$\dim_{\text{cyc}}(V_i) \leq \max\Big(0, \dim_{\text{cyc}}(V) - 3\Big)$$

For a 4-VAS, we have either

- V' is a 2-VASS $\xRightarrow{\text{Blondin et al., 2021.}}$ $\text{poly}(n)$ length bound
- V' consists of two components with $\dim_{\text{cyc}}(\cdot) \leq 1 \implies \dim_{\text{cyc}}(V') \leq 2 \xRightarrow{\text{Proposition 15}}$ $\text{poly}(n)$ length bound.

Proposition 15. There is a polynomial H such that every geometrically 2-dimensional d-VASS is length-bounded by a 2-VASS with H-amplification.

Corollary 16. Reachability in 4-VAS is in **PSPACE** under binary encoding and is **NL**-complete under unary encoding.

5-VAS

I. Reduce Actual Dimension.

V' is a $(d-2)$-VASS with

$$\dim_{\mathrm{cyc}}(V') \leq \dim_{\mathrm{cyc}}(V) - 1.$$

II. Reduce Compopnent-Dimension.

$V' = (V_1)\mathbf{0}(V_2)$ with

$$\dim_{\mathrm{cyc}}(V_i) \leq \max\big(0, \dim_{\mathrm{cyc}}(V) - 3\big)$$

For 5-VAS, we have either

$$\begin{cases} V' \text{ is a 3-VASS} \implies \text{Elementary length bound} \\ V' \text{ is a 5-VASS with} \dim_{\mathsf{com}}(V') \leq 2 \implies \text{Elementary length bound} \end{cases}$$

Corollary 17. Reachability in 5-VAS is in **ELEMENTARY**.

Conclusion

Main Results for Fixed-Dimensional VAS Reachability.

	1	2	3	4	5	$d \geq 6$
Unary	[?, **NL**]	[?, **NL**]	**NL**-complete*	**NL**-complete*	[**NL**, 3-**EXPSPACE**]*	in F_{d-2}*
Binary	**NP**-complete	[**NP**, **PSPACE**]	[**NP**, **PSPACE**] $*\dagger$	[**NP**, **PSPACE**] $*\dagger$	[**PSPACE**, 3-**EXPSPACE**]*	

$[\mathcal{C}_1, \mathcal{C}_2]$ means that the exact complexity is unknown but is $\mathcal{C}_1$-hard and in $\mathcal{C}_2$.

*: The result established in this work.

†: Reachability in 3 and 4-VAS under binary encoding are recently shown to be **PSPACE**-complete in [Kamiński and Lasota, 2026].

Open Problems.

- Is $(d+3)$-VAS length-bounded by a d-VASS with suitable amplification for $d \geq 3$?
- What is the exact complexity of reachability in binary 2-VAS?
- Is reachability in 5-VAS in **PSPACE**?